Discussion Papers in Economics

ANGLO-DUTCH, SPLIT-AWARD SPECTRUM AUCTIONS WITH A DOWNSTREAM MARKET

By

Parimal Bag
(University of Surrey)

Paul Levine
(University of Surrey)

&

Neil Rickman
(University of Surrey and CEPR)

DP 05/07

Department of Economics
University of Surrey
Guildford
Surrey GU2 7XH, UK
Telephone +44 (0)1483 689380
Facsimile +44 (0)1483 689548
Web www.econ.surrey.ac.uk
ISSN: 1749-5075

Anglo-Dutch, Split-Award Spectrum Auctions with a Downstream Market*

Parimal Bag
University of Surrey

Paul Levine
University of Surrey

Neil Rickman
University of Surrey and CEPR

October 13, 2007

*This paper has been written as a contribution to the EU funded project End-to-End Reconfigurability, phase II (E^2R II). The authors acknowledge the feedback from the E^2R II consortium members, especially University of Surrey colleagues Klaus Moessner and Duminda Thilakawardana.

Abstract

Treating spectrum of different bandwidths as essentially distinct inputs needed for possibly different types of services has formed the core of spectrum analysis in academic research so far. New technological advances, such as cognitive radio, now allow us to move away from this inflexibility and to open up the new possibility of making different spectrum bands compatible. Spectrum, it is envisaged, is to become *divisible and homogeneous*. Auctions for this case have not been previously analyzed. By suitably adapting the Anglo-Dutch spectrum auction of Binmore and Klemperer (2000) and the split-award procurement auction of Anton and Yao (1989) and combining the adapted versions, we set out an 'Anglo-Dutch split-award auction' for divisible and homogeneous radio spectrum. An important feature of the game is a post-auction stage where the firms who have acquired some spectrum compete in the production of radio services. The equilibrium of the complete information game is completely characterized and important differences with the procurement auction highlighted. Finally, we compare the performance of our auction mechanism with a complete information form of the Binmore – Klemperer mechanism.

JEL Classification: L10, L50, L96

Keywords: radio spectrum, spectrum trading, imperfect competition.

Contents

1 Introduction

The study of Flexible Spectrum Management, typified by the E2R programme, focuses on technical mechanisms to increase the efficiency of spectrum usage. The pursuit of *technical efficiency* has been parallelled by a world-wide shift towards the greater reliance of market mechanisms to achieve a parallel objective of *allocative efficiency.* This paper examines one such mechanism - the spectrum auction - which has been increasingly relied upon in the United States, the United Kingdom and elsewhere in Europe.[1]

In this paper we consider a particular form of auction mechanism that is particularly relevant for the new telecommunications services sector in the EU, namely the spectrum allocation among a small number of firms, who then compete in the downstream market for the delivery of various services. While our focus is on maximizing license revenues, the spectrum allocation mechanism we are going to propose starts from a base level importance attached to consumer welfare. Simply put, this means each firm in the downstream market will have a minimum market share and a monopoly situation does not arise.

While the important theoretical contribution by Milgrom (2000) has rightly placed auctions in the limelight in engineering spectrum allocations, and quite a few papers deal with spectrum auctions,[2] only three papers explicitly address auctions with some downstream market interaction possibility (Jehiel and Moldovanu, 2001; Janssen, 2006; Janssen and Karamychev, forthcoming). None of these papers consider, however, the textbook style imperfect competition oligopoly model that perhaps captures better the market interaction. To fill this gap in the literature, we allow for a quantity setting oligopoly competition in the market interaction stage after spectrum has been allocated.

Our basic auction-and-market-interaction model extends the pure auction model

[1]See Bourse et al (2007) for a comprehensive overview of EU spectrum policy.

[2]See Jehiel and Moldovanu (2003) for a comprehensive survey of how auctions have been used by various countries for spectrum allocations. The survey also offers some guidance to related papers by academic researchers. See also Klemperer (2002).

– an *Anglo-Dutch* auction – initially proposed for the sale of the British 3G telecom licenses. This auction method is outlined in detail by its two main advocates, Professor Ken Binmore and Professor Paul Klemperer in Binmore and Klemperer (2002). Briefly, the Anglo-Dutch auction first selects $n+1$ bidders out of m bidders, $m > n+1$, using an ascending bid auction (alternatively known as *English auction*) for the right to further bid in a second-round auction for n licenses. The price thus rises until $n+1$ bidders remain. In the second round, each remaining bidder submits a sealed bid at or above the price at which the first-round bidding had stopped; the top n bidders in the second-round bidding win the licenses and pay either their respective bids or the n-th highest bid.[3] When the winning bidders pay their respective bids, the procedure is known as *first-price auction*; when the bidders pay n-th highest bid, it is a simple extension of what is known as *second-price auction* (or Vickrey auction).

Binmore and Klemperer highlighted three aspects as the auction's principal objectives: efficiency of spectrum assignment, promotion of competition, and realization of the full economic value.[4] Efficiency meant awarding the licenses to bidders with the best business plans, which in turn was expected to translate into relatively higher valuations for the licenses. It is well known, however, that efficiency, revenue maximization and promotion of competition often do not go hand-in-hand; some compromise is expected. Both Binmore–Klemperer recommended auction and the auction mechanism in this paper follow this principle of balance of objectives.

At this stage we like to note some special features of the particular auction that was originally proposed for the British telecom licenses, because some of these features motivate us to adapt/modify the Anglo-Dutch auction the way we do (as

[3] Actually, there was also a third stage involved because of the heterogeneity of the licenses. The third stage was another auction known as 'simultaneous ascending price auction,' used in the sale of spectrum by the US FCC. In this auction, multiple bidders bid for various units of the licenses in rounds. The bidding would go on until there are two final rounds of bidding in which no new bids are submitted for any of the licenses. The licenses are then disposed to the highest bidders in the last active phase, with each winning bidder paying what they had bid in the last active phase.

[4] A total of four licenses had been auctioned off.

detailed further below). First of all, the licenses were heterogenous and had different amounts (and types) of spectrum associated with each license, and which were chosen by the licensing authority (although in imperfect anticipation of the likely market demands that are going to prevail). This heterogeneity – both in quantity and quality – is likely to involve some inevitable inefficiency given that various bidders have diverse interests. As for volumes of spectrum per license, any ad hoc specification by the licensing authority would leave unexploited many other possible specifications that the bidders as a whole (or even the majority of the bidders) might have strictly preferred.[5] We therefore aim to disentangle the "lumpiness" of the licenses by treating spectrum as a perfectly divisible commodity and letting the bidders themselves express their preferences for the continuum of this divisible unit. We do this simplification by dispensing with the heterogeneity of the spectrum's quality; we will assume all spectrum is identical in that they are inputs to generate the same (or similar) type of service. Thus, in the product market we consider interactions between firms producing a *homogeneous good*. Finally, we model the spectrum commodity as an essential ingredient to produce the final output (or service) by assuming a Leontief-type production technology (see section 2 for details).

The case of divisible, homogeneous spectrum has not been previously analyzed. The two aspects – homogeneity and divisibility – are of course closely related. Treating spectrum of different bandwidths as essentially two distinct inputs needed for possibly different types of services formed the core of spectrum analysis in academic research so far. But with the new technological possibility of making different spectrum bands compatible[6] This new and the possibility of servicing more customers with better spectrum capacity, with the latter becoming increasingly feasible due

[5]This, in itself, need not be a bad thing for the licensing authority. As it is commonly known, given the number of licenses to be awarded, restricting to exogenous, and possibly heterogenous, quantity allocations in licenses (as opposed to endogenously determined allocation per license) may limit collusion among bidders; see Anton and Yao, 1989; Klemperer, 2002; Janssen, 2006.

[6]In particular, these new possibilities are associated with cognitive radio–see Qinetic (2006) and the E2R programme.

to allowance for *spectrum trading* by various firms, makes the analysis of allocation of a continuum of spectrum shares very much relevant. If spectrum can be traded by license holders according to individual demand conditions, then consideration of an initial variable allocation of spectrum based on bidders' individual bids is not an unrealistic mechanism.

Given divisible spectrum, rather than a discrete number of (possibly heterogenous) licenses, we will consider the split-award auction mechanism that is often used for government procurement of a fixed volume of certain services and formally analyzed by Anton and Yao (1989). In the split-award procurement auction, bidders submit sealed bids for their respective shares in the service contract and the government (or the auctioneer) chooses the split that maximizes the sum total of bids. We adapt this split-award mechanism in one of the stages comprising the spectrum assignment problem.

In summary, our spectrum assignment game involves fours stages. The government has a fixed amount of spectrum to be allocated. In stage 1, a given number of potential firms participate in an ascending price auction to win a minimum pre-specified amount of the available spectrum and to be able to further bid for additional spectrum in a subsequent sealed-bid auction. All but two firms are eliminated in the ascending price auction and the remaining two firms pay the final dropout price, earn the pre-specified minimum spectrum and then proceed to stage 2. In stage 2, the two firms submit bids for various shares of the remaining spectrum, and in stage 3 the government chooses the split that maximizes the total bids. Finally in stage 4, the two firms compete in the service provision market and their outputs are constrained, through a Leontief production technology, by the amount of spectrum won in the ascending price and split-award auctions. We call this procedure the *Anglo-Dutch, split-award auction.* More formal description of this game appears in section 3.

The differences between the Anglo-Dutch auction of Binmore and Klemperer and the auction mechanism we propose in this paper are several. First, the spectrum licenses in our context allow for a continuum of shares as opposed to a discrete number of licenses and this requires a different auction technique in the form of split awards. Second, the ascending bid auction of Binmore and Klemperer ensures a minimal starting bid for the eventual $n + 1$ bidders who participate in the sealed-bid auction stage for n discrete licenses. In contrast, the ascending bid stage of our auction selects two firms for a guaranteed minimal amount of spectrum each with the possibility of additional spectrum; importantly, the option values of acquiring additional spectrum at some price do get reflected in the bidders' strategies during the ascending price auction, thus the ascending bid stage serves both for screening and surplus extractions. The minimal spectrum awarded to the two firms through the ascending bid auction also ensure that the market never degenerates into a monopoly. The basic principles behind Binmore–Klemperer method and our auction method are, however, similar – generate high overall revenues for the government and ensure some necessary competition in the downstream market for consumer welfare.

The rest of the paper is organized as follows. Section 2 presents the downstream market interaction. In section 3, we formally outline the spectrum assignment game, followed by an analysis of the game in section 4. In section 5, we illustrate our results from section 4 using a simple linear market demand and quadratic cost curves. Section 6 concludes.

2 The Anglo-Dutch, Split-Award Spectrum Auction Game

The spectrum manager will award a given volume of spectrum, x, in two phases: first, a number of potential competitors will be screened down to two who both are given a threshold amount of spectrum z; then the same two firms compete for any share of the remaining spectrum, $x - 2z$. The initial screening and then disbursement of spectrum follows a combination of Anglo-Dutch auction mechanism of Binmore–Klemperer and the split-award auction mechanism of Anton–Yao. The precise mechanism, to be referred as Γ, is as follows:

Stage 1. m interested firms, who all know each others' cost functions and the common market demand function, take part in an ascending bid auction. Bids rise continuously until all but two firms drop out. Denote the bid at which $(m-3)$rd firm drops out by ξ. The two surviving firms, denoted as firm 1 and firm 2, pay each ξ for a guaranteed amount of spectrum, z, and then they proceed to compete in stage 2.

Stage 2. Firms 1 and 2 submit sealed bids $(p_1(\alpha), p_2(\alpha))$, $\forall \alpha \in [0,1]$ where α denotes firm 1's share of $x - 2z$ and $1 - \alpha$ denotes firm 2's share.

Stage 3. The government chooses the split α^* to maximize the total bids:

$$g(\alpha) \equiv p_1(\alpha) + p_2(\alpha). \tag{1}$$

Any ties resulting in the highest total bids will be broken using uniform probability. The bidders pay their submitted bids for the split chosen by the government.

Stage 4. Given their spectrum capacity constraints $r_1 = z + \alpha^*(x - 2z)$ and $r_2 = z + (1 - \alpha^*)(x - 2z)$, the two firms then go into the market competition stage and produce the spectrum services, q_1 and q_2. The market price is determined according to a downward-sloping inverse market demand function $P = D(Q)$ where $Q = q_1 + q_2$. The two firms finally realize their profits less respective sum total of bids in the two auctions, $\xi + p_1(\alpha^*)$ and $\xi + p_2(\alpha^*)$. ||

We need to solve the extensive form game Γ backwards. Initially we focus on the last three stages of the game for any two firms selected in stage 1. The analysis of this subgame will follow closely Anton and Yao's analysis of a split-award procurement auction; our split-award spectrum bidding game is a mirror image of Anton and Yao's procurement auction. Restriction to two firms for the stage 2 bidding, is to keep the analysis tractable: with two firms, each firm's bidding depends only on its own share as it also uniquely determines the other firm's share; for more than two firms each firm's bidding strategy would depend not only on its own share but also on other firms' shares. We do not believe the extra complications due to more than two firms would alter the main message of our paper qualitatively.

The procedure for calculating the equilibrium bids are set out in Anton and Yao (1989), and adapted to our spectrum sale setting in section 4. These form a Nash equilibrium at the bidding stage of the game.

The important feature of the second stage bidding game is the minimal information requirement on the part of the government, who needs to observe only the bids by the two firms. Note also that the government's objective function is a balance between revenue maximization and guaranteeing consumer welfare through some minimal competition in the downstream market. This is achieved by deciding that only $x-2z$ spectrum will be distributed through the split-awards auction; by giving at least z spectrum to each of two firms, the downstream market never becomes a monopoly.

3 Stage 4: The Downstream Market

We consider a single local market with a small number of firms with some market power and providing a homogeneous service at a market price P. In this section, the number of firms, n, is unspecified. In the sections to follow, we will consider $n = 2$.

Firm i produces output q_i, $i = 1, 2, \cdots, n$ and the total output $Q = \sum_{i=1}^{n} q_i$. The inverse demand curve is given by $P = D(Q)$; $D'(Q) < 0$. Units of output are customer-minutes of some service requiring radio channels as an input.

To keep the analysis simple we consider a decreasing returns to scale Leontief technology

$$q_i = \min[r_i, \psi(\mathsf{y})] \tag{2}$$

where y is a vector of other inputs such as base-stations and labour. Thus output of firm i is constrained by the volume r_i of radio channel licences issued to firm i. We assume that firm i has acquired the spectrum at a price p_i. Costs include a set-up cost F_i, so total costs are given by

$$TC_i(q_i, \xi, p_i, F_i) = F_i + \xi + p_i + C_i(q_i), \tag{3}$$

where $C_i(\cdot)$ depends exogenously on the given factor prices for inputs y with $C_i'(\cdot) > 0$, $C_i''(\cdot) > 0$.

In a constrained Cournot-Nash equilibrium, firm i faces a capacity constraint $q_i \leq r_i$. Given fixed costs and previous bid prices, it then maximizes gross profits at stage 4 given by

$$\pi_i = Pq_i - C_i(q_i) \tag{4}$$

subject to the capacity constraint, taking the output of all other firms, $\sum_{j \neq i}^{n} q_j = \tilde{q}_i$, as given. Notice that licensing costs, $\xi + p_i$, are now part of fixed costs and only affect the firm's participation constraint at stage 1.

To carry out this constrained optimization programme, define the Lagrangian

$$L_i = \pi_i + \lambda_i(r_i - q_i) + \mu_i \pi_i \tag{5}$$

where $\lambda_i \geq 0$ and $\mu_i \geq 0$ are Lagrangian multipliers associated with constraints $r_i \geq q_i$ and $\pi_i \geq 0$ respectively. The Kuhn-Tucker first-order conditions for a maximum are:

$$(1 + \mu_i)\left[P + q_i D'(Q) - C_i'(q_i)\right] = \lambda_i \tag{6}$$

$$\mu_i \pi_i = \lambda_i(r_i - q_i) = 0. \tag{7}$$

The left-hand-side of equation (6) is defined as $(1+\mu_i)f(q_i)$. At stage 4 we must have that $\pi_i > 0$ so that $\mu_i = 0$. Further, if $f(q_i) > 0$ then $\lambda_i > 0$, which in turn implies $q_i = r_i$ so that firm i uses all its acquired channels producing at full-capacity. If on the other hand $f(q_i) = 0$, then the capacity constraint need no longer be binding so that $q_i \leq r_i$ and there might be spare radio channels and capacity.[7] In this last case, q_i is given by (6) with $\mu_i = \lambda_i = 0$.

[7] It is not possible to have $f(q_i) < 0$ at an optimal choice of q_i because (6) would be violated.

The Cournot-Nash equilibrium of the production stage of the game (subsequently referred to as programme **CNE**) is then given by solving

$$\begin{aligned} [P + q_i D'(Q) - C_i'(q_i)] &= \lambda_i, \quad i = 1, 2, \ldots, n \\ \lambda_i (r_i - q_i) &= 0, \quad i = 1, 2, \ldots, n \\ \pi_i &= P q_i - C(q_i) \quad i = 1, 2, \ldots, n \\ Q &= \sum_{i=1}^{n} q_i \\ P &= D(Q) \end{aligned}$$

which gives $2n + 2$ equations in q_i, λ_i, Q and P, given r_i, ξ and p_i, $i = 1, 2, \ldots, n$.

4 Stages 2 and 3: Equilibrium bids

In this section, we characterize various properties of equilibrium bidding in stage 2 of the game Γ. Since a number of our results draw upon Anton–Yao's equilibrium characterization, most of the results will be stated without formal proofs.

4.1 Monotone bidding

The firms' duopoly profits (excluding the spectrum costs) for the split, α, will be denoted by $\pi_i(\alpha)$, $i = 1, 2$. We assume that $\pi_1'(\cdot) \geq 0$, $\pi_1''(\cdot) \leq 0$ and $\pi_2'(\cdot) \leq 0$, $\pi_2''(\cdot) \leq 0$. That is, having a greater share of the available spectrum can do no harm to a firm's profits and the marginal impact (on profits) is weakly decreasing.[8]

We are going to present some equilibrium properties of the bid functions $(p_1(\alpha), p_2(\alpha))$. We will assume that the firms adopt, in equilibrium, *(weakly) monotone* bid functions: $p_1^{*\prime}(\alpha) \geq 0$, $p_2^{*\prime}(\alpha) \leq 0$ ((*) to denote equilibrium). We assume monotone bidding to make the equilibrium analysis tractable. But there is also the obvious intuition in favor of monotone bidding. Define total revenue by

$$g(\alpha) = p_1(\alpha) + p_2(\alpha). \tag{8}$$

[8]This assumption may not be as innocuous as it may sound. With more spectrum available, a firm may not be able to convince its rival that it will not expand output along the reaction curve.

Lemma 1. *(Price equivalence) Suppose that* $(p_1^*(\alpha), p_2^*(\alpha))$ *is a Nash equilibrium in (weakly) monotone bidding, and let* g^* *be the associated total revenue to the government. Then, the equilibrium bids satisfy* $g^* = p_1^*(1) = p_2^*(0)$.

Proof. Suppose the claimed equality fails to hold. Then one of the bid prices must be smaller than g^* as g^* is the total revenue to the government at the optimal choice. If $p_2^*(0) < g^*$, then the outcome $\alpha = 0$ is not optimal for the government.

Let $\epsilon = g^* - p_2^*(0) > 0$, and consider firm 1's bid as follows:

$$p_1(\alpha) = \begin{cases} p_1^*(\alpha) - \frac{\epsilon}{2}, & \text{if } \alpha > \alpha^* \\ p_1^*(\alpha) - \frac{\epsilon}{3}, & \text{if } \alpha = \alpha^* \\ 0, & \text{if } \alpha < \alpha^*, \end{cases} \tag{9}$$

where α^* is an optimal choice for the government in the original equilibrium. In fact, given that the government's optimal choice can be random (with ties in total bids over multiple values of α), we take α^* to be the best ex-post equilibrium choice from the point of view of firm 1.

Below we will argue that given firm 2's equilibrium bidding, $p_2^*(\alpha)$, the modified bidding strategy $p_1(\alpha)$ in (9) will imply α^* is the unique new optimal choice for the government at which firm 1 receives strictly higher payoff than his best ex-post payoff in the original equilibrium.

First observe that $p_1^*(0) = p_2^*(1) = 0$, as a firm cannot be expected to submit a positive bid for a zero share in the split. Next, $p_1^*(\alpha^*) = g^* - p_2^*(\alpha^*) \geq g^* - p_2^*(0) = \epsilon$, given that $\alpha = 0$ is not an optimal choice originally and the equilibrium bid functions are weakly monotonic. This implies $p_1(\alpha) = p_1^*(\alpha) - \frac{\epsilon}{2} > p_1^*(\alpha^*) - \frac{\epsilon}{2} > 0$ for $\alpha > \alpha^*$ (the first inequality follows because $p_1^*(.)$ is monotonically increasing), and similarly $p_1(\alpha^*) = p_1^*(\alpha^*) - \frac{\epsilon}{3} > 0$.

Now, using (9) obtain

$$p_1(\alpha) + p_2^*(\alpha) = \begin{cases} p_1^*(\alpha) + p_2^*(\alpha) - \frac{\epsilon}{2}, & \text{if } \alpha > \alpha^* \\ p_1^*(\alpha) + p_2^*(\alpha) - \frac{\epsilon}{3}, & \text{if } \alpha = \alpha^* \\ p_2^*(\alpha), & \text{if } \alpha < \underline{\alpha}^*, \end{cases} \tag{10}$$

which implies α^* is the new unique optimal choice for the government for the modified bidding $p_1(\alpha)$ by firm 1.

Finally, firm 1 is strictly better off compared to his best ex-post outcome in the original equilibrium (p_1^*, p_2^*), because all that is different in this new unique optimal choice of α^* is that firm 1 pays a strictly lower price $p_1^*(\alpha) - \frac{\epsilon}{3}$ (rather than $p_1^*(\alpha)$) for the same split $\alpha = \alpha^*$. Thus, p_1^* is not a best response – a contradiction. Hence, it must be that $p_2^*(0) = g^*$. The case for $p_1^*(1) < g^*$ is analogous. **Q.E.D.**

4.2 Efficiency and Monopoly Outcomes

Staying with the duopoly case, the analogy of joint production costs in the procurement auction of Anton and Yao is joint operating profits

$$B(\alpha, z, x) = \pi_1(\alpha, z, x) + \pi_2(\alpha, z, x) \tag{11}$$

where $\pi_i(\alpha, z, x)$ are operating profits for firm $i = 1, 2$ at stage 4 given a split $[\alpha, 1-\alpha]$ of the spectrum offered at stage 2, $x - 2z$, and x is the total spectrum available for the market. Then for $z \in [0, \frac{x}{2}]$ we have

$$\pi_1(1, z, x) > 0\,,\ \pi_2(1, z, x) \geq 0 \text{ with } \pi_2(1, 0, x) = 0 \tag{12}$$

$$\pi_1(0, z, x) \geq 0 \ \text{ with } \pi_1(0, 0, x) = 0\,,\ \pi_2(0, z, x) > 0 \tag{13}$$

$$p_1(0, z, x) = p_2(1, z, x) = 0. \tag{14}$$

To ease the notation in what follows we abbreviate $B(\alpha, z, x)$ to simply $B(\alpha)$ and similarly $p_i(\alpha, z, x)$ to $p_i(\alpha)$ for $i = 1, 2$ (as in the previous section).

Then the case where all the spectrum is available at stage 2, $z = 0$, is the spectrum auction counterpart of the model of Anton and Yao. In what follows let firm 1 be the low cost firm with $c_1 < c_2$ and $0 < d_1 < d_2$ for the decreasing returns to scale case, $d_1 < d_2 < 0$ for the increasing returns to scale case. Then $B(1) > B(0)$ and the counterpart of proposition 1 in Anton and Yao with $z \geq 0$ is:

Proposition 1. *Suppose x and $z \in [0, \frac{x}{2}]$ are given. Suppose also that $B(1) > B(\alpha)$*

for all $\alpha \in [0,1)$. *Then* $\alpha = \alpha^* = 1$ *is the unique bidding equilibrium that satisfies*

$$\begin{aligned} g^* &= B(1) = p_1^*(1) = p_2^*(0) \\ \Pi_1^* &= B(1) - B(0) > 0 \\ \Pi_2^* &= \pi_2(1, z, x) \geq 0 \text{ for } z \geq 0 \end{aligned}$$

where net[9] *and operating profits,* Π_i *and* π_i *respectively, are related by*

$$\Pi_i(\alpha) = \pi_i(\alpha) - p_i(\alpha)\,; \;\; i = 1, 2. \tag{15}$$

It follows from definitions (8), (11) *and* (15) *that the sum of net profits is given by*

$$\Pi_1(\alpha) + \Pi_2(\alpha) = \pi_1(\alpha) + \pi_2(\alpha) - (p_1(\alpha) + p_2(\alpha)) = B(\alpha) - g. \tag{16}$$

4.3 Implicit Price Collusion and Split-Award Outcomes

Now in contrast to Proposition 1, suppose that there exists $\alpha \in [0,1)$ such that Suppose $B(1) < B(\alpha)$. Then the following proposition mirrors the procurement auction of Anton and Yao.

Proposition 2. *Let* $N = \{\alpha \mid B(\alpha) \geq B(1), \alpha \in (0,1)\}$ *be the set of outcomes for which joint operating profits are greater than the monopoly profits of the low cost firm 1. Then* N *is the set of split-award equilibria. These equilibria* $\alpha^* \in N$ *are characterized by:*

$$g^* \in [B(0) + B(1) - B(\alpha^*), B(0)] \tag{17}$$

$$\Pi_1^* \in [B(\alpha^*) - B(0), B(1) - g^*] \tag{18}$$

$$\Pi_2^* \in [B(0) - g^*, B(\alpha^*) - B(1)]. \tag{19}$$

Propositions 1 and 2 completely characterize the equilibria at stage 2 of the game. In Proposition 2, (17) determines the *revenue* which from (16) determines *total net profits.* Then (18) and (19) determine the *division* of total profits between the two bidders. Given net profits and operating profits given at stage 4, the equilibrium bid

[9] Our definition of 'net' profits excludes the bid price paid at stage 1.

prices are then determined and we have therefore characterized for both firms their equilibrium bid prices and net profits at stage 2. Following AY for $B(1) < B(\alpha)$ the equilibria satisfies the following corollary:

Corollary. *Let $\alpha \in N$. Then the minimum for the revenue, and the maxima for joint total profits and individual total profits, occur at the split-award outcome for which joint operating profits are maximized.*

5 Stage 1: The Initial Ascending Bid Auction

Consider $m > 2$ interested firms who take part in an ascending bid auction. We assume a quadratic cost function

$$C_i(q_i) = c_i q_i + d_i q_i^2, \quad i = 1, 2 \tag{20}$$

and that cost parameters can be ranked so that

$$c_1 < c_2 < c_3 \cdot \cdot \cdot < c_m \tag{21}$$

$$d_1 < d_2 < d_3 \cdot \cdot \cdot < d_m. \tag{22}$$

Then for both increasing returns $d_i < 0$ and decreasing returns $d_i > 0$ we have that

$$C_1'(q) < C_2'(q) < C_3'(q) \cdots < C_m'(q) \quad \text{for all } q \geq 0\ ;$$

$$C_1(q) < C_2(q) < C_3(q) \cdots < C_m(q) \quad \text{for all } q \geq 0.$$

Denote by $\Pi_i(\alpha^*(i, j))$ where $i \neq j$ and $i, j \in (1, 2, \cdot \cdot \cdot m)$ the *net profit* to firm i earned in the downstream market when i and j enter stage 2 bidding and subsequently win a share $\alpha^*(i, j)$ of the spectrum on offer. What dropout bid a firm chooses at any point in the ascending bid auction would depend on its expectation of the sequence of dropouts in the game from that point onwards. In particular, if a firm i were to continue in the auction (rather than drop out immediately), it must choose to do so expecting to win the auction with another firm j and making non-negative net profits, $\Pi_i(\alpha^*(i, j)) \geq 0$, from stage 2 onwards. This requires solving the extensive form game of dropouts explicitly. Given the clear ranking of firms in terms of both marginal and total costs, there will be a similar clear ranking in terms of of profits:

$$\begin{aligned} \Pi_i(\alpha^*(i,j)) &> \Pi_j(\alpha^*(i,j)), \quad \text{for any } \{i,j\} \text{ pair such that } i<j\ ; \\ \Pi_i(\alpha^*(i,j)) &> \Pi_j(\alpha^*(i,j)) > \cdots > \Pi_k(\alpha^*(i,k)), \quad \text{for any } \{i,j,\cdots,k\} \\ & \qquad\qquad \text{such that } i<j<\cdots<k\ . \end{aligned} \tag{23}$$

Given (23), it can be verified that the following is an equilibrium sequence of dropout bids:

$$\begin{aligned} & b_m < b_{m-1} < \cdots < b_3 < b_2 < b_1, \\ \text{with} \quad & b_m = \Pi_m(\alpha^*(1,m)) \\ & b_{m-1} = \Pi_{m-1}(\alpha^*(1,m-1)) \\ & \cdots \\ & b_3 = \Pi_3(\alpha^*(1,3)) \\ & b_2 = \Pi_2(\alpha^*(1,2)) \\ & b_1 = \Pi_1(\alpha^*(1,2)). \end{aligned} \tag{24}$$

In fact, in *any equilibrium* firms 1 and 2 will always win and their dropout bids are given uniquely as in (24). The reasoning for the dropout decisions is standard – same as why truthful bidding is an equilibrium in a private-value, English auction.

Thus, firms 1 and 2 win the ascending bid stage and both pay the entry fee

$$\xi = \Pi_3(\alpha^*(1,3)), \tag{25}$$

and proceed to stage 2 bidding.[10]

[10] We assume that firm 3 will drop out at $\Pi_3(\alpha^*(1,3))$ even though it can continue up to just below $\Pi_2(\alpha^*(1,2))$ and then drop out because the latter strategy does not yield firm 2 any chance of a win in which it would make positive profit. In fact, under incomplete information about rival firms' costs (as opposed to the complete information assumption made in this paper), a firm will never bid above its true valuation (i.e., profit).

6 Comparison with Binmore–Klemperer Auction

How does our auction mechanism perform relative to Binmore– Klemperer mechanism? In one respect, such comparison may not even be appropriate due to an important difference. The division of Binmore–Klemperer mechanism into ascending bid auction and sealed-bid auction was designed in view of the *incomplete information* among the bidders about each others' valuations: ascending bid auction ensured that the top few bidders would participate in the sealed-bid stage and thus ensuring the spirit of efficiency, whereas sealed-bid stage kept alive the chance of a relatively 'weak' type firm to be the ultimate winner and thereby prompting the 'stronger' types to bid aggressively; also sealed-bid would have minimized the chance of collusion. In a *complete information* setup (where firms know each others' cost functions and thus valuations from various split awards), which we assume (partly for tractability reasons), Binmore–Klemperer mechanism reduces to a solitary sealed-bid auction; their ascending-bid stage no longer gives any added value as only the 'best' two firms (best in terms of cost efficiency, given a ranking of firms is possible) would be in a position to win any spectrum, with the third-best firm disciplining the top two firms' bids in the sealed-bid stage.[11]

Despite removing one main ingredient of Binmore–Klemperer mechanism – the incomplete information among bidders – a complete-information analogue of Binmore–Klemperer mechanism may still offer an important benchmark with respect to which our mechanism can be assessed. With this objective, below we analyze a reduced-form Binmore–Klemperer mechanism, which is a sealed-bid first-price auction involving top three firms who each bid for one of two identical licenses, with each license awarding half the overall amount of spectrum, x.[12]

[11]The idea is that the dropout bid of a weaker firm in the ascending-bid stage can always be improved by a stronger firm, because the stronger firm can expect to achieve at least as much profit as a weaker firm in the subsequent sealed-bid stage.

[12]Allowing the licenses to be heterogeneous would have required analysis of a much more complicated auction. Our objective is to offer at least one comparison to assess our mechanism. Also, we restrict to first-price auction for the reduced Binmore–Klemperer game, rather than Vickrey-type

By analogy with $\Pi_i(\alpha^*(i,j))$ defined in the previous section, denote by $\pi_i(i,j)$, where $i \neq j$ and $i,j \in \{1,2,3\}$, the *operating profit* to firm i from operation in the downstream market when i and j win the licenses (that award each half the spectrum). We assume the following inequalities (without verification) that are quite intuitive:

$$\pi_1(1,2) > \pi_2(1,2) > \pi_3(1,3) \tag{26}$$

$$\pi_1(1,3) > \pi_3(1,3) \tag{27}$$

$$\pi_1(1,3) > \pi_2(2,3) \tag{28}$$

$$\pi_2(2,3) > \pi_3(2,3) > \pi_3(1,3). \tag{29}$$

Proposition 3. *Consider a sealed-bid first-price auction involving firms 1, 2 and 3 who bid for one of two identical licenses, with each firm having an 'exit option' if its bid is the joint lowest with at least another firm's bid. When there is a tie and none of the firms whose bids are tied exercise their exit options, then each tied firm is chosen to be a winner with equal probability. Suppose*

$$\pi_1(1,2) \;\; > \;\; \pi_3(2,3) \tag{30}$$

$$\textit{and} \qquad \pi_2(1,2) \;\; > \;\; \pi_3(2,3). \tag{31}$$

Then

$$(b_1,\ b_2,\ b_3) = (\pi_3(2,3),\ \pi_3(2,3),\ \pi_3(2,3)) \tag{32}$$

is an equilibrium, in which firm 3 exercises its 'exit option' while firms 1 and 2 win a license each and pay their bids.

Proof. First we claim that in any equilibrium firm 1 must win a license. Suppose not, so that firms 2 and 3 win licenses. This implies $b_3 \leq \pi_3(2,3)$, because otherwise firm 3 will incur losses that it can avoid by submitting zero bid. But then firm 1 can deviate from its original bid b_1 by submitting $\hat{b}_1 = \pi_3(2,3) + \epsilon < \min\{\pi_1(1,2), \pi_1(1,3)\}$ where $\epsilon > 0$ and is sufficiently small; $\pi_3(2,3) + \epsilon < \pi_1(1,2)$

auction, because our split-award mechanism involves pay-your-own-bid by the firms.

given (30), and $\pi_3(2,3)+\epsilon < \pi_1(1,3)$ given (28) and the first inequality of (29). Thus, by deviating firm 1 would win a license and make a positive profit; a contradiction.

Similarly, firm 2 also must win a license in any equilibrium. Suppose not, so that firms 1 and 3 win licenses. This implies $b_3 \leq \pi_3(1,3)$, because otherwise firm 3 will incur losses that it can avoid by submitting zero bid. But then firm 2 can deviate from its original bid b_2 by submitting $\hat{b}_2 = \pi_3(1,3)+\hat{\epsilon} < \min\{\pi_2(1,2), \pi_2(2,3)\}$ where $\hat{\epsilon} > 0$ and is sufficiently small; $\pi_3(1,3) + \hat{\epsilon} < \pi_2(1,2)$ given the second inequality in (26), and $\pi_3(1,3) + \hat{\epsilon} < \pi_2(2,3)$ given (29). Thus, by deviating firm 2 would win a license and make a positive profit; a contradiction.

Next we claim that in equilibrium firm 1 and firm 2's bids must be tied: $b_1 = b_2$. If not, suppose $b_1 > b_2$. Then clearly firm 1 can slightly lower its bid and still exceed firm 2's bid, thus ensuring that it wins a license and pays less overall and thereby making a higher net profit. On the other hand, if $b_2 > b_1$ then by a similar logic firm 2 can slightly lower its bid, retain its license-winning position and make a higher net profit.

It remains to verify that (32) is indeed an equilibrium. Given that $b_3 = \pi_3(2,3) = b_j$, it is clearly optimal for firm i to bid $b_i = \pi_3(2,3)$ where $i \neq j$ and $i, j \neq 3$ (by (30) and (31)). Also, given $b_1 = \pi_3(2,3) = b_2$, firm 3 does no worse by bidding $b_3 = \pi_3(2,3)$ and subsequently exercising its exit option. Note that if firm 3 were not to exercise its exit option, it may well end up with firm 1 (because of the tie-breaking rule), in which case it would make losses. **Q.E.D.**

Firm 3's bid puts a floor to the top two firms' bids. Actually, firm 3 need not actively participate in bidding. So long as the licensing authority can bring in firm 3 at a "small cost" (by offering some direct inducement) to participate in the sealed-bid auction, firms 1 and 2 will end up bidding $\pi_3(2,3)$ even in the two-bidder auction involving firms 1 and 2.

7 The Case of Linear Demand and Quadratic Costs

We now study the split award and Binmore-Klemperer auctions for particular functional forms for demand and costs. In particulat we assume that $D(Q)$ is linear and $c(q_i)$ quadratic. Thus,

$$P = \begin{cases} a - bQ, & Q < \frac{a}{b}; \\ 0, & Q \geq \frac{a}{b}. \end{cases} \tag{33}$$

We first examine the split award auction for the special case of the game where $z = 0$ so that it is possible for on firm to emerge as the monopoly producer. We first consider a symmetric n-firm equilibrium at stages 2 and 3 and then an asymmetric duopoly.

7.1 Split Award Auction: Symmetric Case for $z = 0$

Let $\Pi_i = \pi_i - p_i$ where π are operating profits excluding spectrum costs. At the production stage the firm then maximizes π_i with respect to q_i. Let π_i^{NE} and π^M be the operating profits per firm for in the CNE and a (one site) monopoly respectively. The following Lemma is crucial for the existence of a split award equilibrium:

Lemma 2. *There exists an amount x of spectrum and a split award such that in the production stage of the game,*

$$\sum_{i=1}^{n} \pi_i^{NE} \quad > \quad \pi_j^M, \quad \textit{for some } j \in \{1, 2, \ldots n\}$$
$$\textit{i.e., for the duopoly case} \quad B(\alpha) \quad > \quad B(1) \textit{ for some } \alpha \in [0, 1). \tag{34}$$

Proof. We prove this for identical firms. First we work out the unconstrained CNE for this case. Each firm then puts $Q = q + \tilde{Q}$ and takes the output of others, $\tilde{Q}$ as given. Maximizing $\pi = Pq - c(q) = (a - b(q + \tilde{Q})q - ca - dq^2$, this gives a reaction function

$$q = \frac{a - b\tilde{Q} - c}{2(b + d)}. \tag{35}$$

In a symmetric CNE, $\tilde{Q} = (n-1)q$ which leads to equilibrium total output

$$Q^{NE} = nq^{NE} = \frac{a-c}{b + \frac{1}{n}(b+2d)} \tag{36}$$

$$P^{NE} = \frac{bc + \frac{a(b+2d)}{n}}{b + \frac{1}{n}(b+2d)}. \tag{37}$$

It follows that as the number of firms gets large, price tends to marginal cost. At the other extreme if $n = 1$, we arrive at the monopoly output

$$Q^M = \frac{a-c}{2(b+d)} < Q^{NE}. \tag{38}$$

To prove the Lemma set $x = Q^{NE}$ so that the demand for spectrum in the CNE is just met without any spare capacity. The spectrum manager must compare the total operating profits in this equilibrium with an equal split (since all firms are identical) with that under under monopoly where some spectrum is left spare. Total operating profits are then

$$\pi^M = (a-c)Q^M - (b+d)(Q^M)^2 \equiv \pi^M(Q^M) \tag{39}$$

$$\begin{aligned} n\pi^{NE} &= n[(a-c)q^{NE} - (bQ^{NE}q^{NE} + d(q^{NE})^2) \\ &= (a-c)Q^{NE} - \left(b + \frac{d}{n}\right)(Q^{NE})^2 \\ &= \pi^M(Q^{NE}) + d\left(1 - \frac{1}{n}\right)(Q^{NE})^2. \end{aligned} \tag{40}$$

It follows that $n\pi^{NE} > \pi^M$ if and only if

$$d > \frac{\pi^M(Q^M) - \pi^M(Q^{NE})}{\left(1 - \frac{1}{n}\right)(Q^{NE})^2} \equiv F(d). \tag{41}$$

$F(d) > 0$ for $n \geq 2$, because $\pi^M(Q)$ reaches its maximum at $Q = Q^M$. (41) says that the Lemma holds if there exists a fixed point for the function $F(d)$, say d^* and if $d > d^*$. No doubt, with a bit of effort, one can show generally that $F(d)$ has a fixed point. Below we produce a numerical example for which this is the case.

In the figure we have set $b = c = 1$, $a = 2$ and $n = 2, 3, 5$. The corresponding fixed points are d_n. In each case for $d > d_n$ the Lemma holds. Note that d_n is an increasing function of n. The intuition is that as N increases then so do the monopoly operating profits, thus increasing the extent of diminishing returns that would make the gains from output sharing dominate in a CNE.

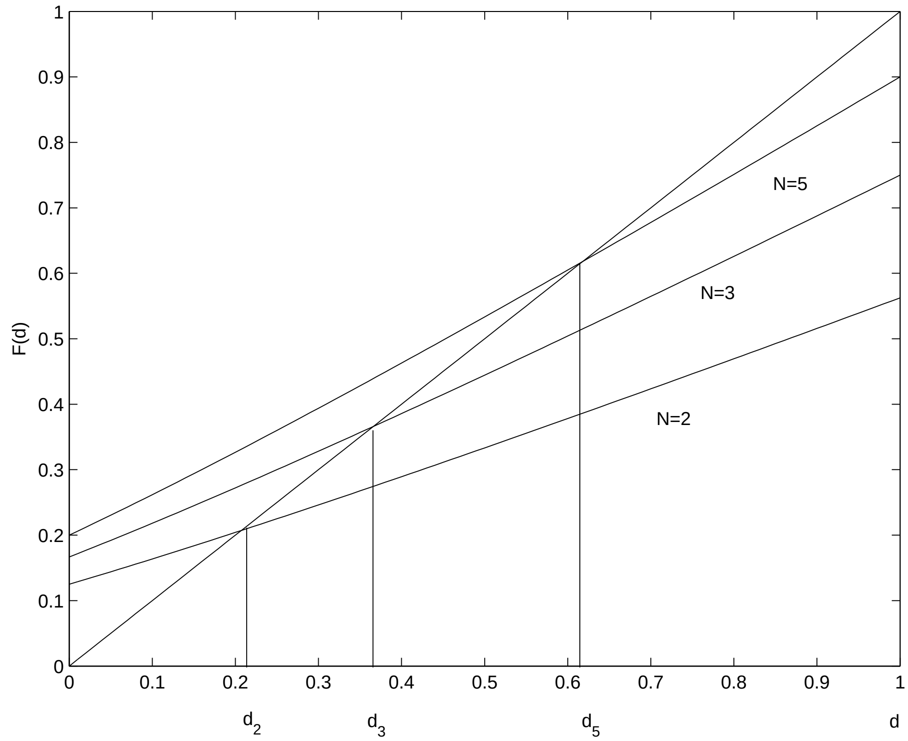

Figure 1: **Fixed Points of** $F(d)$

7.2 Split Award Auction: Asymmetric Duopoly for $z = 0$

Now we have to solve for the full CNE allowing for the capacity constraints set out in section 2. Solving this problem and noting that bids at zero output satisfy $p_1(0) = p_2(1) = 0$ we have reaction functions.

$$0 \leq q_1 = \min\left[\alpha x,\, (a - c_1 - bq_2)/(2(b + d_1))\right] \tag{42}$$

$$0 \leq q_2 = \min\left[(1-\alpha)x,\, (a - c_2 - bq_1)/(2(b + d_2))\right]. \tag{43}$$

Then unused spectrum, $\hat{q}_1$ and $\hat{q}_2$ respectively, is given by

$$\hat{q}_1 = \alpha x - q_1 \tag{44}$$

$$\hat{q}_2 = (1-\alpha)x - q_2. \tag{45}$$

Figures 2 and 3 illustrate these results.[13] In these figures we put x equal to the total output in the unconstrained NE so that the spectrum manager is allowing for

[13] We set $a = 3$, $b = 1$, $c_1 = 0.5$, $c_2 = 0.85$, $d_1 = 0.1$, $d_2 = 0.2$.

this output with an appropriate split. Joint operating profits for any (α, x, z) are now denoted by $B(\alpha, x, z)$. From figure 2 there is a collusive split award equilibrium at $\alpha^* = 0.94$ and in figure 2 we see that firm 1 has spare capacity at this split. There are other possible non-collusive split award equilibria at which $B(\alpha, x, 0) \geq B(1, x, 0)$. In this example these exist for $\alpha \in [0.9, 1]$. *The existence of spare spectrum in equilibrium highlights an important difference between the procurement and spectrum sale problems. In the former the scale of output is given, but in the latter the quantity of service provided depends on the equilibrium split.*

7.3 Split Award Auction: Asymmetric Duopoly for $z > 0$

With $z = 0$ we have seen that a split-award equilibrium only exists if there is *sufficiently decreasing returns to scale.* Now consider the case of $z > 0$ in which case the capacities at the production stage of the game are given by

$$r_1 = z + \alpha(x - 2z) \in [z,\, x - z]\,;\ r_2 = z + (1 - \alpha)(x - 2z) \in [z,\, x - z]. \tag{46}$$

First consider the unconstrained Nash equilibrium which is the outcome at this stage of the game provided that $x - z$ is sufficiently large. From the reaction functions (42) and (43) this is given by

$$q_1^{NE} = \frac{2(a - c_1)(b + d_2) - b(a - c_2)}{4(b + d_1)(b + d_2) - b^2} \tag{47}$$

$$q_2^{NE} = \frac{2(a - c_2)(b + d_1) - b(a - c_1)}{4(b + d_1)(b + d_2) - b^2}\,. \tag{48}$$

The question we now pose is whether *there exists some $\alpha \in [0, 1)$ such that for some values of x and $z < \frac{x}{2}$, and parameters a, b, $c_i > 0$ and $d_i \geq 0$ or $d_i \leq 0$ we have that $B(\alpha, z, x) \equiv \pi_1(\alpha, z, x) + \pi_2(\alpha, z, x) > B(1, z, x)$?* The key to showing that this in fact can happen is to assume that x and z are chosen so that

$$q_2^{NE} < z \leq r_2 = z + (1 - \alpha)(x - 2z) \tag{49}$$

$$q_1^{NE} > x - z \geq r_1 = z + \alpha(x - 2z). \tag{50}$$

Then firm 2 is unconstrained and firm 1 is constrained for all $\alpha \in [0, 1]$. It follows from the reaction function of firm 2 that the equilibrium is given by

$$q_1 = z + \alpha(x - 2z) \tag{51}$$
$$q_2 = \frac{(a - c_2 - bq_1)}{2(b + d_2)} \tag{52}$$
$$P = a - b(q_1 + q_2) \tag{53}$$
$$\pi_i = (P - c_i)q_i - d_i q_i^2 , \quad i = 1, 2. \tag{54}$$

To make the analysis more tractable, first consider the case of constant returns to scale, $d_i = 0$, $i = 1, 2$. Then joint operating profits and the price are given by

$$B(\alpha, x, z) = \frac{q_1}{2}(P - 2c_1 + c_2) + \frac{(P - c_2)(a - c_2)}{2b} . \tag{55}$$

Hence as α and therefore q_1 rise we have

$$\frac{dB(\alpha, x, z)}{d\alpha} \propto \frac{dB(\alpha, x, z)}{dq_1} = \frac{1}{2}(P - 2c_1 + c_2) + \left[\frac{q_1}{2} + \frac{a - c_2}{2b}\right]\frac{dP}{dq_1} . \tag{56}$$

The first term on the right-hand-side is positive and represents the effect of increasing output by giving the more efficient firm 1 more capacity. The second term is negative and represents the effect on joint profits of a fall in the price. A little algebra gives

$$\frac{dB(\alpha, x, z)}{d\alpha} \propto \frac{dB(\alpha, x, z)}{dq_1} = \frac{1}{2}(2(c_2 - c_1) - bq_1). \tag{57}$$

Substituting for q_1 from (51) we arrive at the following proposition:

Proposition 4. *For constant returns to scale, a collusive split-award equilibrium exists at $\alpha = \alpha^*$ where*

$$\alpha^* = \frac{2(c_2 - c_1) - bz}{b(x - 2z)}, \tag{58}$$

provided (49) *and* (50) *hold and c_2, c_1, b, z and x are such that $\alpha^* \in (0, 1)$.*

7.4 Numerical Illustration and Comparison of Auctions

In Figures 4 to 6 we illustrate these results for the Split Award Auction. We first assume constant returns to scale ($d_1 = d_2 = 0$) and as before choose $a = 3$, $c_1 = 0.5$,

$c_2 = 0.85$, $b = 1$. Then $q_1^{NE} = 0.63$ and $q_2^{NE} = 0.40$. Then choose $z = 0.44$ and $x = 1.01$ so that $q_2^{NE} < z, q_1^{NE}$ and $x > 2z$. Then the conditions for proposition 4 are satisfied and from (58), $\alpha^* = 0.2$ which is confirmed in the figure. Figure 5 now allows for decreasing returns to scale by putting $d_1 = 0.1$ and $d_2 = 0.2$ with remaining parameters unchanged. Now the firms benefit by shifting capacity more to the more efficient firm and the split award equilibrium rises to $\alpha^* = 0.53$.

Figure 6 adds revenue to figure 4 with and without the initial bidding stage. In figure 6 we set the marginal cost of the third firm at $c_3 = 0.9$, compared with $c_2 = 0.85$. The bid as defined in section 5 is $\xi = 0.005$ which sees the regulator acquiring over half the joint operating profits. However this results is critically dependent on how contestable the market is and in particular how close c_3 is to c_2.

c_3	$\Pi_3(\alpha^*(1,3))$	$\alpha^*(1,3)$	SA Revenue	BK Revenue
0.90	0.0025	0.6	0.8500	0.6546
0.95	0.0012	0.8	0.8487	0.6067
1.00	0.0004	0.9	0.8479	0.5704
1.05	0	1.0	0.8475	0.5375

Table 1. Comparisons of Revenues from Split Award (SA) and Binmore-Klemperer (BK) Auctions

Table 1 shows what happens as the market becomes less contestable as c_3 rises and we compare the revenues for the Split Award and Binmore-Klemperer auctions, the latter with an even split. For the former we see that the initial bid price falls sharply until at $c_3 = 1.05$ it disappears altogether and then stage 1 adds no more to the revenue. From (17), (25) and (32) 1 the total revenue split for the split award (SA) auction and Binmore-Klemperer (BK) auctions are given by

$$\textbf{SA Revenue} = \underbrace{\Pi_3(\alpha^*(1,3))}_{\textbf{Stage 1 Revenue}} + \underbrace{B(0) + B(1) - B(\alpha^*(1,2))}_{\textbf{Stage 2 Revenue}} \quad (59)$$

$$\textbf{BK Revenue} = \pi_3(2,3) \quad (60)$$

It is difficult to compare these revenues analytically, but table 1 provides a numerical comparison for this particular set of parameters with $x - 2z$ chosen so as to result in a split award outcome. The results suggest that the *SA auction results in a higher revenue than the BK auction.* For this example this is not the result of the first stage bid of the SA which does not increase revenue a lot. Rather it comes about because of the ability of the auctioneer to extract a sizable proportion of total operating profits in a collusive split equilibrium at stage 2 of the game.

The higher revenue however comes at a cost. In the SA equilibrium in our example with constant returns to scale it is required that the efficient firm 1 is constrained for all possible splits but the inefficient firm 2 is not (conditions (49) and (50)) In the collusive equilibrium we then have more spectrum allocated to the unconstrained inefficient firm, some of which is subsequently unused. This means that the BK auction with an equal split imposed results in higher output and a lower price. To assess the welfare implications of this we calculate the consumer surplus (CS) given by

$$\mathrm{CS} = \int_P^a \left(\frac{a-P}{b}\right) dP = \frac{1}{b}\left[\frac{1}{2}(a^2+P^2) - aP\right] \tag{61}$$

c_3	CS^{SA}	R^{SA}	PS^{SA}	CS^{BK}	R^{BK}	PS^{BK}
0.90	0.6700	0.8500	0.0022	0.7057	0.6546	0.2465
0.95	0.6700	0.8487	0.0035	0.7057	0.6067	0.2944
1.00	0.6700	0.8479	0.0043	0.7057	0.5704	0.3307
1.05	0.6700	0.8475	0.0047	0.7057	0.5375	0.3636

Table 2. Comparisons of Welfare in Split Award (SA) and Binmore-Klemperer (BK) Auctions

Figure 7 compares the price and consumer surplus for the SA and BK auctions using (61). Table 2 provides a welfare breakdown for various marginal costs of the third firm, c_3, into consumer surplus (CS), revenue (R) and producer surplus (PS) defined as total operating profits for the two firms minus total spectrum costs (equals revenue). These results highlight a further important difference between the mechanisms: the revenue in the BK auction is totally dependent on the efficiency of the third firm whereas in the SA auction even in the absence of a competitive third firm the auctioneer can still extract substantial revenue at stage 2. Total social welfare (CS+R+PS) is higher for the BK auction unless we allow for a shadow cost of public funds arising from the distortionary impact of alternative forms raising revenue from taxes (see Laffont and Tirole (1993)). If however the auctioneer is only concerned with consumer surplus plus revenue, the SA auction is clearly preferable in our example.

8 Conclusions

Treating spectrum of different bandwidths as essentially two distinct inputs needed for possibly different types of services has formed the core of spectrum analysis in academic research so far. The E2R approach to spectrum allocation is to move away from this inflexibility and to open up the new technological possibility of making different spectrum bands compatible. Spectrum, it is envisaged, is to become *divisible and homogeneous*. Auctions for this case have not been previously analyzed.

Our analysis has shown that the split-award spectrum auction for homogeneous divisible spectrum closely resembles the mirror image of the Anton–Yao procurement auction, but with one important difference: whereas for procurement, the scale of the the project (i.e., the output in the downstream market is given), for the allocation of spectrum the total output of radio services using spectrum as an input depends on the strategic interaction of the producers. In particular we have found split-award equilibria in which spectrum is under-utilized despite being chosen to be just sufficient to service the unconstrained Nash equilibrium in the final market stage of the game.

We have completed the characterization of the equilibria of both the splt award (SA) and Binmore-Klemperer (BK) auction for the complete information case. We have shown that a split-award collusive equilibrium exists for certain certain configurations of parameters. The comparison with the BK auction suggests that the split award auction can deliver both significantly higher revenue and higher revenue plus consumer surplus. Furthermore the revenue in the BK auction is totally dependent on the competition from the most efficient third firm that drops out in the bidding stage.

There a number of interesting areas for future research including: a generalization of the analysis to the case of incomplete information; a study of other government objectives that include considerations of expected consumer and producer surpluses in addition to revenue; the consideration of substitution in the production stage between spectrum and other inputs and finally the introduction of a further fourth stage of bilateral trading in the game.

References

[1] Anton, J. and Yao, D.A., 1989. Split awards, procurement and innovation. Rand Journal of Economics 20, 538-552.

[2] Back, K. and Zender, J.F., 1993. Auctions of divisible goods: On the rationale for the treasury experiment. Review of Financial Studies 6, 733-764.

[3] Binmore, K. and Klemperer, P., 2002. The biggest auction ever: The sale of the British 3G telecom licenses. Economic Journal 112, C74-C96.

[4] Bourse, D. and Ballon, P. and Cordier, P. and Delaere, S. and Deschamps, B. and Granblaise, D. and Moessner, K. and Simon, O., 2007. The E2R II flexible spectrum management (FSM) - technical, business and regulatory perspectives. E2R White Paper.

[5] Demange, G., Gale, D. and Sotomayor, M., 1986. Multi-item auctions. Journal of Political Economy 94, 863-872.

[6] Janssen, M.C.W., 2006. Auctions as coordination devices. European Economic Review, 50, 517-532.

[7] Janssen, M.C.W. and Karamychev, V.A., 2006. Selection effects in auctions for monopoly rights. Forthcoming, Journal of Economic Theory.

[8] Jehiel, P. and Moldovanu, B., 2001. Auctions with downstream interaction among buyers. Rand Journal of Economics 31, 768-791.

[9] Jehiel, P. and Moldovanu, B., 2003. An economic perspective on auctions. Economic Policy, April issue, 269-308.

[10] Klemperer, P., 2002. How not to run auctions: The European 3G mobile telecom auctions. European Economic Review, 46, 829-845.

[11] Laffont, J-J and Tirole, J. (1993). A Theory of Incentives in Procurement and Regulation. MIT Press.

[12] Milgrom, P.R., 2000. Putting auction theory to work: the simultaneous ascending auction. Journal of Political Economy 108, 245-272.

[13] Qinetic (2006). Cognitive Tadio Technology. A Study for Ofcom.

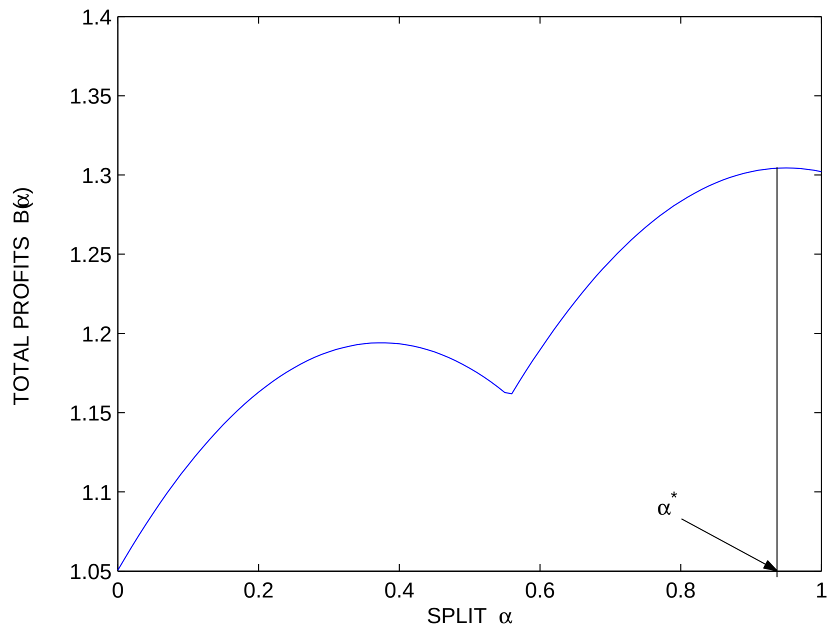

Figure 2: $z = 0$ **and Decreasing Returns to Scale: Total Operating Profits** $B(\alpha, x, 0)$

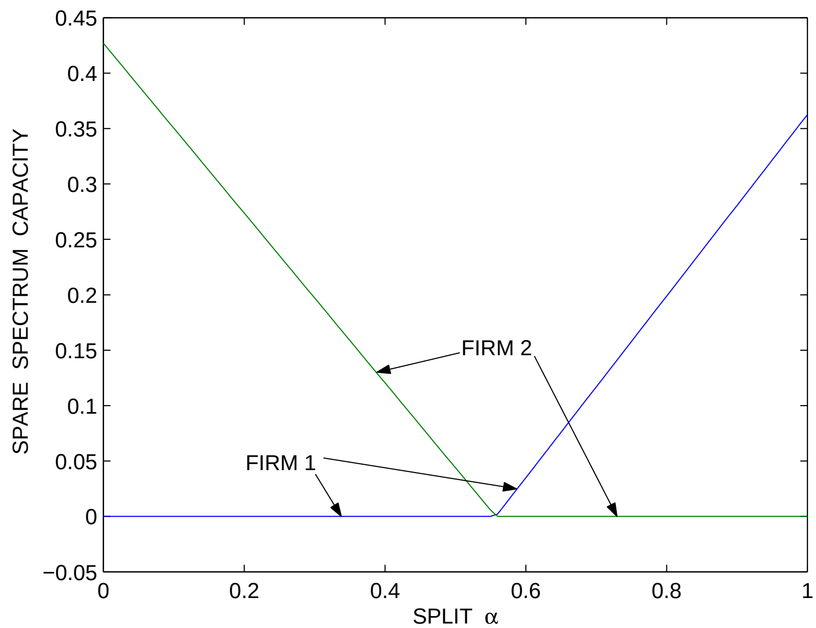

Figure 3: **z=0 and Decreasing Returns to Scale: Spare Spectrum Capacity,** $\hat{q}_1$ **and** $\hat{q}_2$**.**

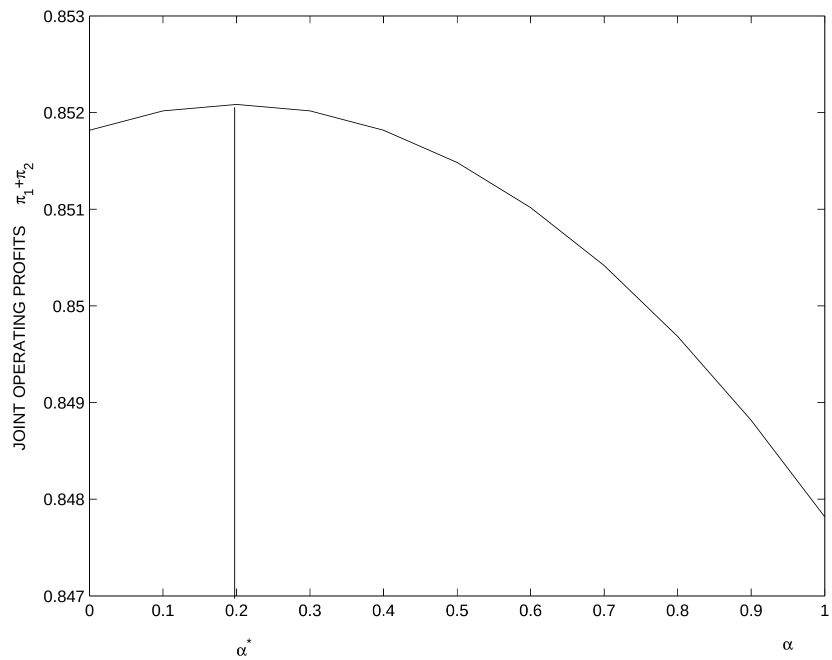

Figure 4: $z > 0$ **and Constant Returns to Scale: Total Operating Profits** $B(\alpha, x, z)$

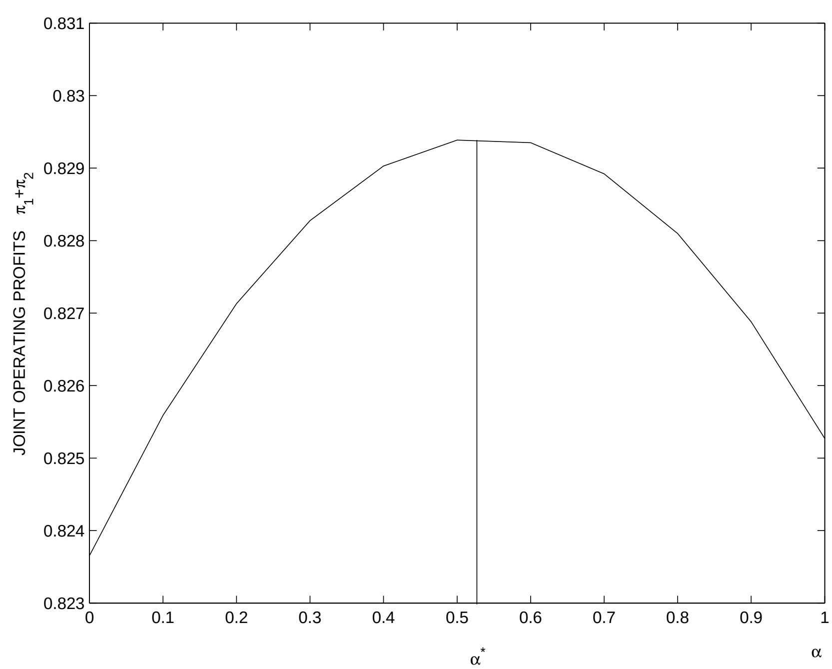

Figure 5: $z > 0$ **and Decreasing Returns to Scale: Total Operating Profits,** $B(\alpha, x, z)$

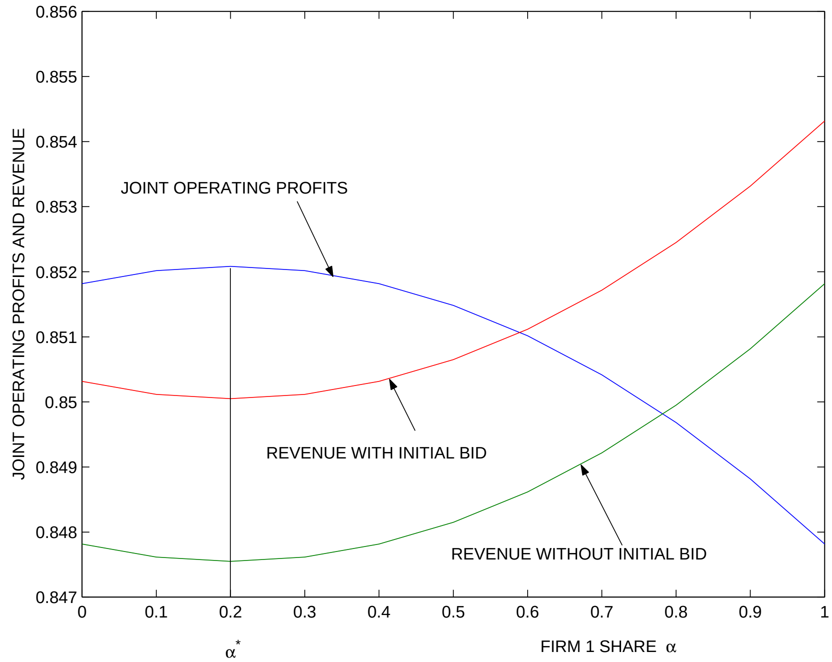

Figure 6: $z > 0$ **and Constant Returns to Scale: Total Operating Profits and Revenue** $B(\alpha, x, z)$ **and** g**.** $c_3 = 0.9$

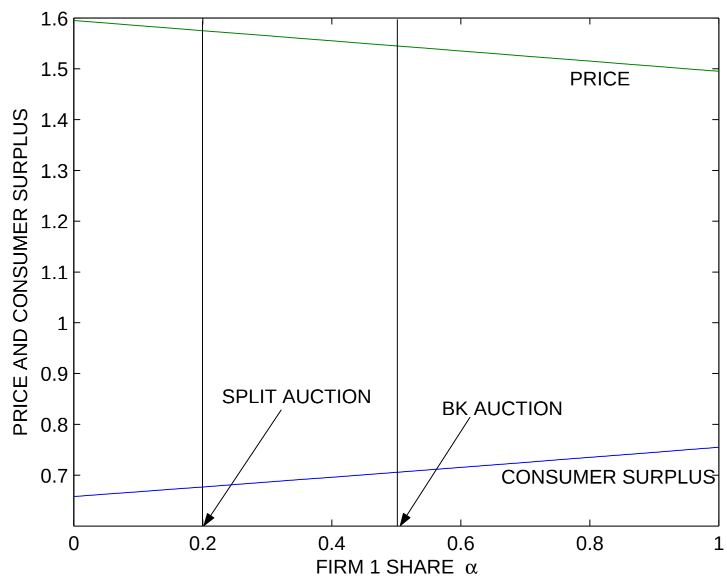

Figure 7: $z > 0$ **and Constant Returns to Scale: Price and Consumer Surplus.** $c_3 = 0.9$